Things of science

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PROBABILITY

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This unit consists of two cubes, one white and one yellow and 20 disks, ten white, six yellow and four blue, and this explanatory booklet.

"Heads or tails?", and a coin is flipped. How many times have you heard this? This simple action, lightly used to make simple decisions, such as to walk or ride or which team plays first, is a part of a complex mathematical theory known as probability.

Probability theory received attention from mathematicians early in the 17th Century when Blaise Pascal, a French mathematician, scientist and religious philosopher, along with Pierre Fermat, another French mathematician, undertook the study of mathematical analysis of games of chance.

In the years following Pascal's study, probability theory became increasingly important and today applications are often made by the physicist who computes the probable path of an atomic particle. Or, a chemist who deals with chance variations that arise in the measurement of the decay rate of radioactive substances. In more recent extensions of numerical analysis

to the field of geology, statistical analysis along with computer programs, provides a greater range of ways for treating field observations.

Other examples of the applications of probability may be found in studies of inheritance of traits by the offspring of parents possessing certain observable traits, in field tests of varying amounts of fertilizer in agricultural studies, or in the research and development connected with preparation and testing of a new drug.

In probability we attempt to *estimate* the value of a measure or a quantity from the best information available at the moment and use that value in prediction of future events.

For example, we may toss a penny 50 times and observe the number of times the penny falls heads and likewise, the number of times the penny falls tails. From this observed ratio of heads and tails we may then estimate the number of heads which will appear if the penny is thrown 100 times, 200 times or 1,000 times. Thus, if the penny falls heads 21 times out of 50, we may estimate that it will fall heads twice the number of times in 100 tosses, or 42; four times as many in 200 tosses or 84, etc.

Later work with mathematical models will also indicate an *expected* value for the number of heads and tails appearing in any of the above numbers of throws. While the *actual* number of heads appearing in 100 throws may differ from our predicted number of heads, we have used a process which gives us an *estimate* from the best information available at the moment.

This unit will give you an introduction to the theory of probability and help you understand the broad scope of its applications in our daily lives.

Experiment 1. Empty the contents of your THINGS box and from them select three white disks and two yellow disks. Number your three white disks 1, 2 and 3 with a pencil and the yellow disks, 1 and 2.

Place these five marked disks in the empty box, cover and shake to mix. Without looking inside the box, select two disks.

What are their colors and numbers?

There are ten possible combinations of pairs that you may draw, assuming that each disk has the same chance of being drawn, as shown in the following table:

POSSIBLE PAIRS

1. W_1W_2	5. W_2Y_1	8. W_3Y_1
2. W_1W_3	6. W_2Y_2	9. W_3Y_2
3. W_1Y_1	7. W_2W_3	10. Y_1Y_2
4. W_1Y_2		

What is the chance that both disks are white?

Count the number of pairs where both disks are white. In the table they are numbers 1, 2 and 7. From this, we say that the chance that both disks drawn are white is three out of ten or 3/10.

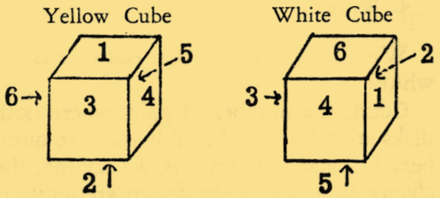
We may, therefore, expect to draw two white disks from a total of three white and two yellow an average of three times in ten draws *in the long run,* or after many trials.

Experiment 2. Make ten consecutive tries. Do your results show three white disks in ten draws? If not, why? (Remember that expectation is based on what will happen *in the long run.*)

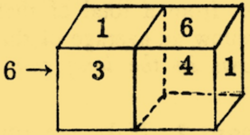
Experiment 3. Return the disks to the box and mix again. Draw again and repeat Experiment 2. What are the chances of drawing two yellows? A white and a yellow?

Experiment 4. Take the yellow and

white cubes and number the faces from 1 through 6 as shown in the diagram below:



When placed in the position indicated below



the sums of all pairs of adjacent faces should total 7, as well as the sum of the faces at opposite ends of the cubes (6 and 1). If not, they have been numbered incorrectly.

Experiment 5. All the combinations that can appear when two cubes are thrown, say for one white and one yellow cube, are shown in the table on the next page. The first number of the paired digits indicates the face appearing on the first

cube, and the second number refers to the second cube:

W Y	W Y	W Y	W Y	W Y	W Y
1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2, 1	2, 2	2, 3	2, 4	2, 5	~~2, 6~~
3, 1	3, 2	3, 3	3, 4	~~3, 5~~	3, 6
4, 1	4, 2	4, 3	~~4, 4~~	4, 5	4, 6
5, 1	5, 2	~~5, 3~~	5, 4	5, 5	5, 6
6, 1	~~6, 2~~	6, 3	6, 4	6, 5	6, 6

For example, what is the probability of getting 8 on one throw of two cubes? From the table given above, you will see there are five ways: (3, 5) (2, 6) (4, 4) (5, 3) (6, 2). Since there are a total number of 36 possible combinations of numbers the probability of getting a total of 8 in one throw is 5/36.

Experiment 6. From the table determine the probability of throwing a double number, two ones, two sixes, etc. (Answer 6/36)

What is the probability that the number on one cube is double the number on the other cube? (Answer 6/36)

Experiment 7. Throw the two cubes a total of 36 times, recording for each throw the value of the number on the white cube and the number on the yellow cube.

Compare these observations with the

table showing the *expected* observations. Do your observations agree with the tabled values? If not, why? The expected values are based on what will happen in the long run.

Experiment 8. From the table determine the probability that the number on the white cube is at least three greater than the number on the yellow cube. (1, 4) (1, 5) (1, 6) (2, 5) (2, 6) (3, 6). (Answer 6/36)

Throw the cubes 36 times and compare the actual results with the expected observation.

Experiment 9. Place the ten white disks, six yellow disks and four blue disks in the box and mix thoroughly by shaking the box. Draw three disks without looking in the box and record the colors of the three selected. This is selection without replacement.

If we attempt to prepare a table showing all the possible combinations of three disks drawn from the box of ten white, six yellow, and four blue, it becomes obvious that observation as a solution to this problem is somewhat laborious.

Let us retain the results of this drawing for computation of the probability of the observed event by another means.

Experiment 10. Probability may be thought of as a ratio, more precisely as the ratio of the number of ways favorable to the occurrence of an event to the number of possible outcomes, both favorable and unfavorable.

Thus, if an event can happen in *m* ways, and fail to happen in *n* ways, the probability of occurrence *p* of the event is,

$$p=\frac{m}{m+n} \qquad (1)$$

and the probability of failure *q*,

$$q=\frac{n}{n+m} \qquad (2) \qquad \text{or}$$

$$p+q=1 \qquad (3)$$

It should be remembered that these probabilities hold only if the $(m+n)$ ways in which the event can occur or not occur are equally likely and mutually exclusive, that is, if one event occurs in one way, the event cannot occur in the other way.

An application of mathematical probability may be made in tossing a single coin where only two outcomes are possible. When we ask the question concern-

ing the probability that the coin will fall heads, applying the equation given above, we see that,

$$p=\frac{1}{1+1}=\frac{1}{2} \qquad (1)$$

where m = favorable outcome, heads, and n = unfavorable outcome, tails. (Note that it is assumed that both events are equally likely.)

The probability of heads appearing on the coin is one-half as shown above and the probability of tails appearing is,

$$q=\frac{1}{1+1}=\frac{1}{2} \qquad (2)$$

Thus the probability that either heads or tails will appear is

$$p+q \text{ or } \frac{1}{2}+\frac{1}{2}=1 \qquad (3)$$

or certainty (neglecting the likelihood that the coin stands on edge).

By tossing the coin thousands of times and keeping a record of the occurrence of heads and tails, we may observe that the ratio of heads to tails approaches 1 : 1.

Other approaches to statistical probability may be made in the same manner

by tossing five coins, ten coins, or any number of coins at the same time and recording the results.

We build a mathematical model to explain the experimental results and thus have a way to estimate the chances that a given event will occur.

It should be remembered that the mathematical model shown for coin tossing indicates what will happen in the long run, and does not indicate certainty of outcome for a single event.

Experiment 11. Toss five pennies 32 times and record the number of heads after each throw, entering the data in the table below.

No. Heads	5	4	3	2	1	0
Observation						

Plot a bar graph showing the number of heads observed.

Experiment 12. In discussing the probability of obtaining one head in one toss of a coin we have limited our mathematical model. Suppose we toss two coins and solve this problem mathematically by our earlier definition of probability.

By writing all the possible combinations which can appear when two coins are tossed:

Observation	First Coin	Second Coin	
1	H	H	(HH)
2	H	T	(HT)
3	T	H	(TH)
4	T	T	(TT)

We see that there are four events which can occur, and by applying formula (1), we see that the probability of occurrence of two heads in one toss of two coins is 1/4.

When we work with only two coins, it is a relatively simple procedure to determine the probability for various outcomes. For obtaining the probability of the appearance of various numbers of heads when greater numbers of coins are tossed, we may make use of Pascal's triangle, an arrangement of the numerical coefficients obtained from the binomial expansion: $(p+q)^n$

$$(p+q)^0 = 1$$
$$(p+q)^1 = p+q$$
$$*\,(p+q)^2 = p^2+2pq+q^2$$
$$**\,(p+q)^3 = p^3+3p^2q+3pq^2+q^3$$
$$(p+q)^4 = p^4+4p^3q+6p^2q^2+4pq^3+q^4$$
$$(p+q)^5 = p^5+5p^4q+10p^3q^2+10p^2q^3+5pq^4+q^5$$

*

$$\begin{array}{r}
(p+q) \\
\underline{(p+q)} \\
p^2+pq \\
\underline{+pq+q^2} \\
p^2+2pq+q^2
\end{array}$$

HH	HT	TT
	TH	

**

$$\begin{array}{r}
(p^2+2pq+q^2) \\
\underline{(p+q)} \\
p^3+2p^2q+pq^2 \\
\underline{p^2q+2pq^2+q^3} \\
p^3+3p^2q+3pq^2+q^3
\end{array}$$

HHH	HHT	TTH	TTT
	HTH	THT	
	THH	HTT	

We may observe on page 14 that the number representing the coefficient, as shown in the small triangle, is equal to the sum of the two numbers above it.

Experiment 13. Compare the frequencies obtained in Experiment 11 with the mathematical expectation shown in Pascal's triangle for tossing five coins 32 times. Remember that the expected numbers of heads should be:

No. Heads	5	4	3	2	1	0
Expectation	1	5	10	10	5	1
Observation						

Do your results agree with mathematical expectation?

Experiment 14. Make a table as in Experiment 13. From Pascal's triangle

Number of Coins	Binomial Coefficients	Number of Different Outcomes
1	1 1	2
2	1 2 1	4
3	1 3 3 1	8
4	1 4 6 4 1	16
5	1 5 10 10 5 1	32
6	1 6 15 20 15 6 1	64
7	1 7 21 35 35 21 7 1	128
8	1 8 28 56 70 56 28 8 1	256
9	1 9 36 84 126 126 84 36 9 1	512
10	1 10 45 120 210 252 210 120 45 10 1	1024

obtain the mathematical expectation if three coins are tossed.

What is the probability of at least two heads as shown in Pascal's triangle? (Answer 4/8)

What are the actual observations?

Experiment 15. What is the probability of getting exactly six heads when ten coins are tossed? (Answer 210/1024)

Experiment 16. By returning to our definition of probability, we may determine the probability, or likelihood of occurrence, of a given event when equal opportunity arises for it to occur or for it to fail to occur, or in cases in which opportunity for failure may exceed opportunity for success.

When we throw a single cube, we observe that the number of equally likely results will be six, the number of faces on the cube. Then referring to formula (1) (Experiment 10), we see that the probability of obtaining a given number, say three, is

$$p = \frac{1}{1+5} = \frac{1}{6}$$

where there is *one* way favorable to the appearance of a three, and there are *five*

ways unfavorable to the occurrence of a three.

Experiment 17. What is the probability of obtaining two fours when two cubes are thrown?

From the table in Experiment 5 we see that there are 36 possible combination of pairs of numbers, but only one that is favorable, that is only one where two fours will appear.

$$\text{Thus, } p = \frac{1}{1+35} = \frac{1}{36}$$

Since the events (rolling the two cubes) are independent of each other, we say that the probability of a four on the yellow cube *and* a four on the white cube is,

$$\text{Probability } (W_4 \text{ and } Y_4) = [\text{Prob } (W_4)][\text{Prob } (Y_4)]$$

$$= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

Experiment 18. What is the probability of throwing a six on the white cube and a five on the yellow cube? Apply the method shown in Experiment 17. (Answer $1/6 \times 1/6 = 1/36$)

Experiment 19. Refer back to Experiment 9. Let us suppose that the three disks drawn were, one white, one yellow and one blue.

Since there are three colors, we see that the probability of drawing a white disk on the first draw is $p=\frac{m}{m+n}$

$$p=\frac{10}{10+(6+4)}, \text{ or } \frac{10}{20}$$

Since there are 10 white, 6 yellow and 4 blue disks.

After drawing the first white, the probability of drawing one yellow is

$$\frac{6}{6+(9+4)}, \text{ or } \frac{6}{19},$$

since this is selection without replacement. And in drawing one blue after drawing one white and one yellow it is

$$\frac{4}{4+(9+5)}, \text{ or } \frac{4}{18}.$$

Then Probability (White, Yellow and Blue), or the probability of drawing one white, one yellow and one blue, in that order, from the box is

$$\frac{10}{20}\times\frac{6}{19}\times\frac{4}{18}, \text{ or } \frac{240}{6840}.$$

Experiment 20. Find the probability of drawing two whites and one blue in that order. (Answer $\frac{360}{6840}$ or $\frac{1}{19}$)

Refer to Experiment 1. We see here that the probability of drawing two white disks from a total of three white disks and two yellow disks is

$$\frac{3}{5}\times\frac{2}{4},=\frac{6}{20} \text{ or } \frac{3}{10}$$

which agrees with our selection of observations.

Experiment 21. Place the four blue disks and the six yellow disks in the box and shake vigorously. What is the probability

(a) If two disks are drawn out of the box without replacement that both are blue? (Answer 12/90)

(b) If two disks are drawn that both are white? (Answer 0)

(c) That the two disks drawn are either both yellow, or one blue and one yellow? (Answer 78/90)

RANDOM NUMBERS

If we wish to draw a sample from a large number of objects and at the same

time give each object an equal chance of inclusion in the sample, a table of random numbers will be quite useful.

In speaking of randomness of numbers, we refer to a series of measures in which there is no pattern of arrangement. For example, research workers and other investigators often find need for a table made up of the ten digits in such fashion that the ten digits occur with equal frequency and in a random manner.

Perhaps one of the simplest ways to prepare a table of random numbers would be to spin a wheel that has ten numbered divisions and record the appearance of the digits when the wheel division number appears by a fixed marker.

If the probability of appearance of each of the ten digits is equal (1/10), we have a process that meets the idea of randomness.

However, this method is both laborious and inefficient. Tables of random numbers are best prepared by electronic devices.

Most tables of random numbers are prepared by arranging numbers at random in columns and rows. The table can be entered at any point and may be read in any direction—up, down, left, or right.

Table of Random Numbers

89326	33491	04617	88092
52171	89301	74066	82717
80748	77622	15779	37361
59807	60562	85747	94028
40422	63035	60344	06883
02627	91576	16781	89184
42096	76920	88864	54164
21871	14672	93362	67981
43845	91838	79574	08003
31674	73729	99315	16699
56753	53158	71872	68153
44708	72952	27048	67887
87112	68614	83073	88794
87316	73087	77135	71883
72976	01868	51667	63279
20523	21584	93712	83654
70603	97122	44978	78028
48410	94516	15427	85323
69788	41758	55004	30992
33884	83655	88345	69602
23053	77480	28683	68324
66660	11057	98849	29499
97247	79368	43710	80365
66300	94385	01717	96191
39860	92127	42588	93307
07223	76264	29148	68652
30786	45403	33782	93424
75275	03080	77653	55430
80166	28017	52611	60012
11317	93109	91857	47904
72927	79021	51571	68825
96086	17329	87959	23727
00448	86828	50552	84832
72894	94716	84622	49771
78590	68615	58113	23727

Experiment 22. Suppose we take our 20 disks and wish to divide them at random into two groups of 10 each. First number the disks 00, 01, 02, and so on with the last number assigned as 19.

A table of random numbers is given on page 20. It consists of four columns of five-digit numbers, with blocks of five numbers per column. Suppose we enter the table in the first column, block two, at the number 02627. Since our disks have been given two-digit numbers, we will make use of only the first two digits of this five-digit number, or 02.

We ignore numbers that are greater than 19 and proceed down the column until we reach the number 07223 and use the first two digits 07. Proceeding downward, using the first two digits of each five-digit number, we select those between 00 and 19, entering the second, third or fourth column as necessary.

If a number once selected reappears in the Table of Random Numbers, ignore it and go on to the next value not yet assigned. When we have reached the first ten numbers below 20, as obtained from the table, we have 10 disks for one group, and the remaining 10 disks form a second group.

We are now ready to extend this process to the selection of groups for experimental and control subjects, where assignment to the group should be made on a random basis.

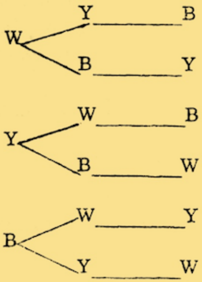
Experiment 23. Select ten single digits from each of the four columns of the Table of Random Numbers. Average the samples and compare their arithmetic means. How do you account for the variation in these means?

Experiment 24. Select a random sample of 30 pages from one of your textbooks. Make a record of the number of pages having either pictures or tables, and then estimate the number of pages in the book having either figures or tables, or both. Compare with the actual count. Would the last 30 pages be as good a sample in general?

ARRANGEMENTS AND COMBINATIONS

In some instances it is useful to know in how many different orders a set of objects can be arranged, as we have shown in the arrangements of two coins tossed at once, or three or more coins tossed simultaneously. The order, or arrangement in some sequence, is called a *permutation.*

For example, select one white disk (W), one yellow (Y) and one blue (B), and determine how many distinguishable ways we can arrange these three disks in a row. A branching tree diagram will help us find this answer:



ARRANGEMENTS

WYB

WBY

YWB

YBW

BWY

BYW

The total number of arrangements, six, may be obtained by multiplying $3 \times 2 \times 1 = 6$, where each arrangement is a permutation. If all objects of the set are used in the arrangement, the number of permutations of *n* objects is then *n*! (read as n factorial), or $n(n-1)(n-2) \ldots . 1$. Thus, we see that from the word "formula," we may obtain $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ arrangements of the seven letters of this word.

Experiment 25. Select four of the disks that were numbered for the experiments with random numbers. Rearrange those four numbered disks in all possible arrangements. Are you able to obtain 24 arrangements?

In certain instances we may wish to know how many different sets of *x* objects can be chosen from a total of *n* objects.

For example, three disks are to be chosen from five disks. In how many ways can they be selected?

Label the disks A B C D E. We can write all the possibilities:

ABC, ABD, ABE, ACD, ACE,
ADE, BCD, BCE, BDE, CDE

We solve this problem by the formula

$$C_{n,x} = \frac{n!}{x!\,(n-x)!} = \frac{5!}{3!\;2!} = 10$$

$C_{n,x}$ is called the number of combinations of *n* objects, taken *x* at each time.

Experiment 26. How many different basketball teams can be formed from 8 players? (Answer $\frac{8!}{5!3!} = 56$)

This brief introduction to probability theory may serve as an indicator of what may be done in making estimates when no exact inference is possible.

University of Tennessee, Knoxville, Tenn. Appreciation is also expressed to Joseph M. Cameron, Silver Spring, Md., for his helpful suggestions.

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